

Fluctuation Relation for Qubit-Calorimetry

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Motivated by proposed thermometry measurement on an open quantum system, we present a simple model of an externally driven qubit interacting with a finite sized, fermion environment acting as calorimeter. The derived dynamics is governed by a stochastic Schrödinger equation coupled to the temperature change of the calorimeter. We prove a fluctuation relation and deduce from it a notion of entropy production. Finally, we discuss the first and second law associated to the dynamics.

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I. INTRODUCTION

How the thermodynamic laws of the macroscopic world transduce and impinge the behavior of structure on the nano- and quantum-level structures, was a question recurrently raised in the past by speculation on the fundamental bounds imposed by physical laws on information processing and transfer [1–4].

Developments of the last decade have demonstrated the experimental feasibility of temperature measurements in nanoscale systems with sub-microsecond time resolution. For example, high sensitivity temperature measurements have been accomplished by embedding a superconductor-insulator-normal metal (SIN) tunnel junction into a radio-frequency (RF) resonant circuit. The temperature of the normal metal side of an SIN tunnel junction is then read by measuring the reflection or transmission coefficient of the LC resonator as a function of both the temperature and of an external current bias [5] (see [7] and the review [9]). SIN tunnel junction thermometers can be employed as fast calorimeters to perform basic studies of the thermodynamics of a mesoscopic nanostructure. As a consequence, questions which were restricted to speculation only a couple of decades ago are now becoming feasible in properly designed experiments.

A fairly recent proposal is the calorimetric measurement of the full distribution of the work done by exerting an external drive on a quantum two level system (qubit) [10]. Let us shortly recall the setup which makes the proposed experiment realistically possible. The qubit is implemented using a solid state electronic circuit [11]. A basic example is a superconducting Cooper pair box (CPB) [11–14] or a transmon qubit [15, 16]. A CPB consists of a superconducting electrode (island) put in contact with a superconducting reservoir. Actual implementations use, e.g., a superconductor-insulator-superconductor (SIS), aluminium-aluminium oxide-aluminium (Al/Al₂O₃/Al), tunnel junction with capacitance C_j . Charges are driven from the reservoir to

the island by a voltage source V between the reservoir and a capacitor C_g connected to the island. If the energy of thermal fluctuations $k_B T$ and the charging (Coulomb) energy of the island are much smaller than the superconducting gap Δ (which is about 1 K for aluminium), all electrons in the island are paired. The experiment thus needs to be performed at temperatures of the order of 0.1 K or below. The effective Coulomb energy of the island becomes $E_C = (2e)^2/2(C_g + C_j)$. Finally, tuning the circuit parameters so that $E_C \gg k_B T$ ensures that only the ground state (no pair in the island) and the first excited state (one pair in the island) have non-negligible probability. A qubit dynamics is thus effectively realized ([17, 18], see also [19] and refs. therein).

A basic calorimeter is formed of a normal metal island with typically 10^9 electrons on it. It is only weakly coupled to the surrounding thermal bath (phonons) at these low temperatures. Calorimetric measurements of the solid state qubit just described can be then performed by connecting the normal metal electrode as a SIN junction thermometer [7, 10] to the resonant circuit. The experimental setup of [10] envisages a drive signal having equal free-energy at the end of the control horizon. Energy conservation then implies that up to boundary terms the work W done on the system is equal to the heat Q dissipated to the environment in the time interval from the beginning of the driving till the end of the equilibration period.

In classical non-equilibrium thermodynamics, a group of relations commonly referred to as fluctuation theorems ([20–25], see also [26–28] for review) links the distribution of the work done on a small system during a transition from an equilibrium state to the free energy of the system at the transition end-states. Classically, the work is a random functional of the protocol, i. e., the sequence of non-conservative forces exerted on the system to drive the transition. From the mathematical slant, fluctuation relations are statements about deviations in the distribution of work around zero. They are therefore obtained by comparing the work distribution during the transition and a second one specified by an appropriately defined time reversed protocol [27].

The extension of fluctuation theorems to the quantum case poses the problem of how to define the work as a functional of the path that the system has followed during the transition. Intense research activity recently focused to determine work for closed and open quantum systems (see [29–38]). For this rea-

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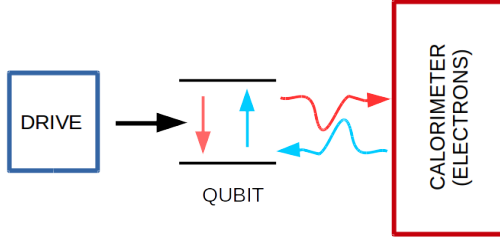


FIG. 1. Pictorial representation of the theoretical model described in sec. II.

son the experiment proposed in [10] stirs interest. It calls for a detailed analysis of how fluctuation relations can be derived and used to achieve a consistent non-equilibrium thermodynamic description of a driven qubit. This is the scope of the present work.

The paper contains two main results. The first is to present a detailed theoretical model for the driven qubit-calorimeter dynamics of [39] (see fig. 1). In particular, in section II we derive the stochastic Schrödinger equation (cf. [40, 41]) describing the evolution of the quantum trajectories followed by a quantum system continuously monitored by an environment, in our case embodied by the calorimeter. In the experimental setup of [39], temperature measurements are possible because the calorimeter can be thought of as a fermion system with a number of degrees of freedom, of the order $\mathcal{O}(10^9)$ larger than the qubit but *finite*. In consequence, the changing temperature effects the coupling between qubit and calorimeter. Hence, the novelty of our model is to derive in addition to the equation for the system evolution its coupling to the evolution law of the temperature of the calorimeter (cf. [39, 42, 43]).

After analyzing the measurement protocol in section III, we derive in section IV the fluctuation relation satisfied by our system under what appears the natural notion of time reversal in our setup (see [10]). We then use the fluctuation relation to identify the entropy production by the driven qubit dynamics. This is the second main result of the paper.

In section V we rephrase our results by deriving the form of the first and second law of thermodynamics. Finally in section VI we shortly describe how to extend our model towards a more realistic situation by taking into account interaction, mediated by phonons, of the calorimeter with the environment.

II. MODELING

The system under consideration consists of a qubit with an energy gap $\hbar\omega_0$ which is driven by an external source and which is coupled to an electron system with finite heat capacity C (see fig. 1). The total system is considered to be governed by the Schrödinger equation with a time dependent Hamiltonian composed of three parts

$$H(t) = H_2(t) + H_e + H_I.$$

Here H_e and H_I denote the free Hamiltonian of the electrons and the interaction Hamiltonian between qubit and electrons. The Hamiltonians

$$H_2(t) = H_2 + H_d(t)$$

acting on the qubit consists of the free Hamiltonian $H_2 = \hbar\omega_0 a^* a$ and a driving Hamiltonian of the form $H_d(t) = \lambda_t(\imath a - \imath a^*)$ varying smoothly in time. Here we write a^* and a for the creation and annihilation operators of the qubit.

During the process we continuously measure the temperature of the electron system, henceforth referring to the electrons as calorimeter. We assume that the number of electrons in the calorimeter is finite but very large and that the temperature is sufficiently small. Then the Sommerfeld expansion (see e.g. appendix C of [44]) may be used to evaluate the average energy of the calorimeter as a function of its temperature. Restricting to the leading order contribution we obtain the relation

$$\Delta(T^2) = \frac{4\mathcal{E}_F}{\pi^2 k^2 \mathcal{N}} \Delta E \quad (1)$$

between the change of the squared temperature $\Delta(T^2)$ and the change of the average energy ΔE , where \mathcal{E}_F is the Fermi energy, k is the Boltzmann constant, and \mathcal{N} is the average number of electrons in the calorimeter.

We assume that the drive changes on much slower time scales than the interaction between the qubit and the calorimeter. Then the usual Born-Markov approximation (see [40, 41] and cf. [45, 46]) yields an effective master equation for the joint system consisting of the qubit and the temperature of the calorimeter. The details of the derivation are outlined in appendix A. For our purposes the master equation is conveniently written as the Itô stochastic differential equation extending the well-known stochastic Schrödinger equation (see [40, 41]):

$$\begin{aligned} d\varphi_t &= \left(\frac{a_t \varphi_t}{\|a_t \varphi_t\|} - \varphi_t \right) dN_t^+ + \left(\frac{a_t^* \varphi_t}{\|a_t^* \varphi_t\|} - \varphi_t \right) dN_t^- \\ &\quad - \frac{\imath}{\hbar} G_t(T_t) \varphi_t dt \\ &\quad + \frac{1}{2} (\gamma_t^+(T_t) \|a_t \varphi_t\|^2 + \gamma_t^-(T_t) \|a_t^* \varphi_t\|^2) \varphi_t dt \\ d(T_t^2) &= \frac{4\mathcal{E}_F}{\pi^2 k^2 \mathcal{N}} \hbar\omega_t \cdot (dN_t^+ - dN_t^-) \end{aligned} \quad (2)$$

with the temperature and time dependent non-Hermitian operator

$$G_t(T) = H_2(t) - \imath \frac{\omega_t}{2} (\gamma_t^+(T) a_t a_t^* + \gamma_t^-(T) a_t^* a_t). \quad (3)$$

In this equation, φ_t and T_t are the pure state of the qubit and the measured temperature at time t , respectively. The operators a_t, a_t^* denote the annihilation and creation operators in the eigenbasis of the qubit Hamiltonian $H_2(t) = H_2 + H_d(t)$ and $\hbar\omega_t$ is the energy gap of $H_2(t)$. The processes N_t^+ and N_t^- are Poisson process with independent increments and rates depending on φ_t and T_t via

$$\begin{aligned} \langle dN_t^+ \rangle &= \gamma_t^+(T_t) \|a_t \varphi_t\|^2 dt \\ \langle dN_t^- \rangle &= \gamma_t^-(T_t) \|a_t^* \varphi_t\|^2 dt. \end{aligned}$$

The rates $\gamma_t^\pm(T)$ are of the form

$$\gamma_t^+(T) = \frac{\gamma}{1 - e^{-\beta(T)\omega_t}}, \quad \gamma_t^-(T) = \frac{\gamma}{e^{\beta(T)\omega_t} - 1}$$

for some constant γ and the inverse temperature $\beta(T) = 1/kT$. The particular form of the rates is not essential for all further derivations, but rather the fact that the rates satisfy the detailed balance condition

$$\gamma_t^+(T) = e^{\beta(T)\omega_t} \gamma_t^-(T).$$

Before we step into the discussion of the general system, it is expedient to take a look at the undriven system, i.e., we assume for a moment $H_2(t) = H_2$ for all times. Let us write (φ, T) for the state of the combined system where the qubit is in the state φ up to a phase and the calorimeter has temperature T . Then, except for an initial period until the first jump, the combined system only attains two states jumping between them back and forth, namely the states (\uparrow, T) for some temperature T and the state $(\downarrow, T + \Delta T)$ with the next higher temperature level $T + \Delta T$. This classical two-level system relaxes to a Gibbs equilibrium distribution. The corresponding Gibbs entropy of the state (φ, T) is given by $\beta(T) \langle \varphi | H_2 | \varphi \rangle + S_{\text{eq}}(T)$, where S_{eq} denotes the *equilibrium entropy* of the calorimeter:

$$S_{\text{eq}}(T) := -\ln \gamma_t^-(T).$$

III. MEASUREMENTS AND THEIR PROBABILITIES

For the system described above we study a concrete measurement protocol specified as follows: We fix a time interval $[t_i, t_f]$ and we assume that the drive vanishes at the beginning and the end of the interval, i.e., $\lambda_{t_i} = 0 = \lambda_{t_f}$. During the interval we proceed as follows:

1. At initial time t_i , we prepare the qubit randomly in one of its eigenstates $\varphi_i = |\uparrow\rangle$ or $\varphi_i = |\downarrow\rangle$. Equivalently, we assume that the qubit is directly measured in this basis.
2. During the time interval we continuously measure the temperature of the calorimeter, which provides a temperature trajectory T_t .
3. At final time t_f , we directly measure the energy of qubit resulting in a state $\varphi_f = |\uparrow\rangle$ or $\varphi_f = |\downarrow\rangle$.

In order to make the distribution of the temperature trajectory explicit, let us fix a trajectory until time $t_i \leq t \leq t_f$. We write $t_i = t_0 < t_1 < \dots < t_n < t_{n+1} = t$ for the partition of the time interval such that the temperature jumps at times t_k for $1 \leq k \leq n$ and remains constant at temperature T_k during the interval (t_{k-1}, t_k) for $1 \leq k \leq n+1$. Then the state of the qubit at time t is given by renormalizing the vector

$$\begin{aligned} \tilde{\varphi}_t &= U_{t,t_n}(T_n) \sqrt{\gamma_n} a_n \dots U_{t_2,t_1}(T_1) \sqrt{\gamma_1} a_1 U_{t_1,t_0}(T_0) \varphi_i \\ &= \left(\prod_{k=1}^n \gamma_k \right)^{1/2} U_{t,t_n}(T_n) \dots a_1 U_{t_1,t_0}(T_0) \varphi_i, \end{aligned} \quad (4)$$

where for sake of brevity we put $\gamma_k := \gamma_{t_k}^+(T_{k-1})$, $a_k := a_t$ if $T_k > T_{k-1}$ and otherwise we put $\gamma_k := \gamma_{t_k}^-(T_{k-1})$, $a_k := a_t^*$. The operators $U_{t,s}(T)$ denote the two-parameter semigroup acting on the Hilbert space of the qubit given by the solution of $U_{s,s}(T) = 1$ and $\frac{d}{dt} U_{t,s}(T) = -\frac{i}{\hbar} G_t(T) U_{t,s}(T)$ for all $s \leq t$. For a trajectory of the full time horizon $[t_i, t_f]$ the conditional distribution of a temperature trajectory $(T_t)_t$ and a final measurement result φ_f is hence given by

$$\begin{aligned} \mathbb{P}(T_t, \varphi_f | \varphi_i) &:= |\langle \varphi_f | \tilde{\varphi}_{t_f} \rangle|^2 \\ &= \left(\prod_{k=1}^n \gamma_k \right) |\langle \varphi_f | U_{t_f,t_n}(T_n) \dots U_{t_1,t_0}(T_0) | \varphi_i \rangle|^2. \end{aligned} \quad (5)$$

IV. FLUCTUATION RELATION AND ENTROPY PRODUCTION

In order to derive a fluctuation relation for the investigated process we choose a natural time reversal. For distinction we henceforth refer to the process described in section II, equation (2), as the forward process. The dynamics of the reversed process is also determined by equation (2) where we replace $G_t(T)$ by the operator $G_t^R(T)$ given by

$$G_{t_f-t}^R(T) = -H_2(t) - i\frac{\omega_t}{2} (\gamma_t^+(T) a_t a_t^* + \gamma_t^-(T) a_t^* a_t).$$

The resulting equation may equivalently be derived by inverting the initial Schrödinger equation of the combined quantum system consisting of qubit and calorimeter and performing the same approximation described in section II. We apply the same protocol to the time reversed process, that is, at time t_i we prepare the qubit, we observe the temperature during the time interval $[t_i, t_f]$, and we measure the qubit directly at final time t_f . Proceeding as in section III we may analogously derive the distribution on the path space for the reversed process.

We now compare a trajectory of the forward process, given by initial and final qubit state φ_i, φ_f and the temperature T_t , with the trajectory of the reversed process that runs backward in time, i.e., the trajectory that starts with a qubit in state φ_f , yields the temperatures T_{t_f-t} , and results in φ_i in the final measurement.

In order to compare the path distribution of the forward process \mathbb{P} and the reversed process \mathbb{P}^R , we may observe the following two facts. First, for each jump in a trajectory of the forward process there is a jump in the opposite direction in the corresponding trajectory of the reversed process. However, the jumping rates are different. Second, the evolution in between temperature jumps is governed by the two-parameter semigroups $U_{t,s}(T)$ with generator $-\frac{i}{\hbar} G_t(T)$ for the forward process and by $U_{t,s}^R(T)$ with generator $-\frac{i}{\hbar} G_t^R(T)$ for the reversed process, respectively. Since $G_{t_f-t}^R(T) = -G_t(T)^*$, the associated semigroups satisfy

$$U_{t,s}^R(T) = U_{t_f-s, t_f-t}(T)^*$$

for all $s \leq t$. We may now compare the expression (4) and (5) for the density of the forward process with the corresponding

expressions for density of the backward process. Gathering the coefficients then immediately yields

$$\mathbb{P}(T_t, \varphi_f | \varphi_i) = e^J \mathbb{P}^R(T_{t_f-t}, \varphi_i | \varphi_f). \quad (6)$$

Here J is the function that, on a path of the forward process, is given by

$$J := \sum_{k=1}^n (\ln \gamma_{t_k}^{x_k}(T_{k-1}) - \ln \gamma_{t_k}^{-x_k}(T_k)),$$

where as before $T_i = T_0, T_1, \dots, T_n = T_f$ is the observed temperature trajectory, $t_i < t_1 < \dots, t_n < t_f$ are the jumping times, and $x_1, \dots, x_n \in \{+1, -1\}$ are the jumping directions $x_k := dN_{t_k}^+ - dN_{t_k}^-$. Due to the detailed balanced condition for the rates, we may write J as

$$J = \sum_{k=1}^n (\Delta S_{\text{eq},k} - \beta_k Q_k),$$

where, for each jump k , we write $\Delta S_{\text{eq},k} := S_{\text{eq}}(t_k, T_k) - S_{\text{eq}}(t_k, T_{k-1})$ for the change of free entropy in the calorimeter, $Q_k := \hbar \omega_{t_k} x_k$ is the heat emitted by the qubit into the calorimeter, and where we put $\beta_k := \beta(T_{k-1})$ if $x_k = 1$ and $\beta_k := \beta(T_k)$ otherwise. (The case distinction is due to the Itô convention in equation (2).)

We may interpret J as an entropy flux from the qubit to the calorimeter. Another contribution to the entropy production is the entropy of the initial preparation and the final measurement of the qubit given by $S_{i/f}(\varphi) = -\ln \mathbb{P}_{i/f}(\varphi)$, where $\mathbb{P}_{i/f}(\varphi)$ is the probability that the state is initially prepared in the state $\varphi \in \{\uparrow, \downarrow\}$ or, respectively, finally observed in φ . The entropy production of the qubit along a trajectory is then given by

$$\sigma := S_f - S_i + J.$$

Equation (6) immediately implies $\langle e^{-\sigma} \rangle = 1$ for the expectation value with respect to the forward process. In particular, it follows $\langle \sigma \rangle \geq 0$.

The derived entropy production can be compared with the approaches to entropy production by Horowitz and Parrondo [36] and by Breuer [47] in the limit of an infinite calorimeter, i. e., when the heat capacity of the calorimeter becomes infinite. In this case the resulting temperature change is negligible during the driving period, and the work and heat measurement strategy needs to be reconsidered. Moreover, in [47] the drive is considered as a small perturbation of the free evolution in the sense that the time dependence of the energy gap of the qubit Hamiltonian is neglected. In this case the entropy flux takes the form

$$J = - \sum_{k=1}^n \beta \hbar \omega_0 x_k = -\beta \hbar \omega_0 (N_{t_f}^+ - N_{t_f}^-),$$

where $\beta = 1/kT$ denotes the inverse temperature and where we assume without loss of generality that the Poisson processes start at $N_0^+ = N_0^- = 0$. Taking the expectation value we recover the entropy flux of [36, 47]. Due to the final measurement of the qubit, the expectation of the entropy production $\langle \sigma \rangle$ is in general larger or equal to the entropy production in [47].

V. FIRST AND SECOND LAW OF THERMODYNAMICS

For the first law we concentrate on the qubit. Suppose at time t the state of the qubit is given by φ_t and the calorimeter has temperature T . Then the expectation value of the energy of the qubit is given by

$$\mathcal{E}(t) := \langle \varphi_t | H_2(t) | \varphi_t \rangle$$

with the qubit Hamiltonian $H_2(t) = H_2 + H_d(t)$. Following [48, 49], the work per unit of time performed on the qubit at time t is given by

$$\mathcal{W}(t) := \left\langle \varphi_t \left| \frac{d}{dt} H_2(t) \right| \varphi_t \right\rangle = \left\langle \varphi_t \left| \frac{d}{dt} H_d(t) \right| \varphi_t \right\rangle.$$

Furthermore, the heat per unit of time emitted by the qubit into the calorimeter at time t is given by

$$d\mathcal{Q}(t) := \hbar \omega_t (dN_t^+ - dN_t^-).$$

Indeed a straightforward computation with (2) yields the stochastic form of the first law of thermodynamics in expectation value:

$$\langle d\mathcal{E}(t) \rangle_{\varphi_t} = \mathcal{W}_t dt - \langle d\mathcal{Q}(t) \rangle_{\varphi_t},$$

where $\langle \cdot \rangle_{\varphi_t}$ denotes the expectation value conditioned on the qubit being in state φ_t .

The second law of thermodynamics may be derived in the limit of an infinite heat capacity. In this case the calorimeter forms an infinite heat bath for the qubit with constant temperature T . The entropy flux J then simplifies to $J = -\beta Q$, where for each trajectory $Q := \sum_{k=1}^n Q_k$ is the total heat emitted by the qubit. The total entropy change in the qubit along a trajectory is hence given by

$$\Delta S := S_f - S_i = \sigma + \beta Q.$$

Since $\langle \sigma \rangle \geq 0$, taking the expectation value yields the familiar form of the second law of thermodynamics,

$$\langle \Delta S \rangle \geq \beta \langle Q \rangle.$$

VI. INCLUDING A PHONON BATH

The model studied so far can be easily extended to the case where the calorimeter is emerged into a phonon heat bath. We assume that the interaction between calorimeter and the phonon bath is weaker but much faster than the interaction between the qubit and calorimeter. Then the energy exchange between calorimeter and phonons only depends on the respective temperatures. For a calorimeter made of ordinary metal, for instance, we have

$$\dot{Q}_{ep} = \Sigma_{ep} (T^5 - T_p^5), \quad (7)$$

where T is the temperature of the electron calorimeter, T_p is the temperature of the phonon bath, Q_{ep} is the heat emitted

by the calorimeter into the phonon bath, and Σ_{ep} is a material dependent constant (see [50]). For materials with a more sophisticated structure, e. g., thin metallic layers, the law (7) may be different (see [51]). For simplicity we assume that the energy exchange is deterministic. Using as before the approximation provided by the Sommerfeld expansion to relate energy changes and temperature changes in the calorimeter via eq. (1), we find that the second equation of (2) has to be replaced by

$$d(T_t^2) = \frac{4\mathcal{E}_F}{\pi^2 k^2 N} \left(\hbar\omega_t(dN_t^+ - dN_t^-) + \Sigma_{ep}(T_p^5 - T_t^5) dt \right).$$

VII. CONCLUSIONS

Summarizing we have established a mathematical model for the dynamics of an externally driven qubit interacting with a continuously monitored, finite size calorimeter. The resulting quantum stochastic evolution (2) extends the usual stochastic Schrödinger equation by incorporating the changing temperature of the calorimeter and its back-action on the measurement process. We derived an explicit expression for the entropy production and demonstrated a corresponding fluctuation relation. As a first application we verified the first law of thermodynamics and proved the second law of thermodynamics in reasonable limiting cases. Finally, we indicated how the model can be extended easily to comprise a common phonon bath.

The model described here is intended as a first approach to the dynamics offering a simple presentation. Experiments suggest that a more realistic model should treat the phonon bath stochastically as well. Such a model is currently investigated numerically [52] in order to analyze the fluctuating thermodynamic quantities beyond the fluctuation relation. In some cases an analytic treatment seems feasible, too. Currently designed experiments address the fluctuation relation derived here. In this way the implications of the finite size of the calorimeter are tested (cf. [53]) and the non-Markovianity of the dynamics can be quantified.

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Appendix A: Separating Time Scales

The fastest time scale in the interaction between qubit and calorimeter is the electron-electron interaction in the calorimeter. This interaction is much faster than any other time-scale involved and does not effect the average energy. Therefore, we treat the self-interaction as an average effect in the van-Hove weak coupling limit [54–56]. On the relevant time scales the calorimeter then can be assumed to be always in a well-defined temperature state characterized by its mean energy.

The second fastest time scale is the interaction between the qubit and the calorimeter. The drive changes on much slower time scale. Then the qubit-calorimeter interaction can be treated by the usual Born-Markov approximation (see [40, Sec. 3.3, Sec. 8.4] or [41, Sec. 3.2]), where the drive is considered to be constant during the interaction. Since we only study energy changes in the calorimeter, we may for this purpose regard the calorimeter as free fermions with corresponding free Hamiltonian $H_e = \sum_k \epsilon_k c_k^* c_k$ and we may take the interaction Hamiltonian of the form

$$H_I = \sum_{k \neq \ell} g_{k,\ell} (a + a^*) c_k^* c_\ell,$$

where the sums are taken over all (pairs of) fermionic modes with corresponding energy ϵ_k and with some interaction coefficients $g_{k,\ell} = \overline{g_{\ell,k}}$.

It is expedient to write the drive in the form

$$H_d(t) = \frac{1}{2} \hbar \omega_0 \tan \theta_t (\imath a - \imath a^*)$$

for some smooth parameter $-\pi/2 < \theta_t < \pi/2$. For this choice of parameters the total Hamiltonian of the qubit $H_2(t) = H_2 + H_d(t)$ is diagonalized by the unitary operator

$$U_t := \begin{pmatrix} \cos(\theta_t/2) & -\imath \sin(\theta_t/2) \\ -\imath \sin(\theta_t/2) & \cos(\theta_t/2) \end{pmatrix}.$$

with generator $H_A(t) := \imath \dot{U}_t U_t^* = -\frac{1}{2} \dot{\theta}_t (a + a^*)$. It is worth noting that the field operator $a + a^*$ commutes with $H_A(t)$ and hence satisfies $U_t(a + a^*)U_t^* = a + a^*$. Passing to the corresponding interaction picture hence does not effect the interaction Hamiltonian. For the derivation of the master equation we treat the drive adiabatically, that is, we neglect the contribution of $H_A(t)$. For the form of the master equation including the computation of the rates we refer to e. g. [40, Sec. 6] (see also [39, 53]), where similar models are treated.

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